Confidence Intervals

- The problem
  - How large are the error bounds when we use data from a sample to estimate parameters of the underlying population?
- Compute confidence intervals for μ
  - when σ^2 is known
  - when σ^2 is unknown

Confidence Intervals

- Precision
- "the range of values in which we can be 95% certain that the "true value" lies"
  - If a RCT is repeated many times you will get many different results
  - these results should cluster around the "true result"
  - the range of results is the confidence interval
- large samples produce narrow confidence intervals
- narrow confidence intervals are precise

Our aim

Suppose an estimate e.g. an estimate for the mean $\bar{x}$ is given

⇕

We want to describe the precision of the estimate

⇕

We do this by giving a range of likely values for the parameter.
Such a range is called confidence interval.

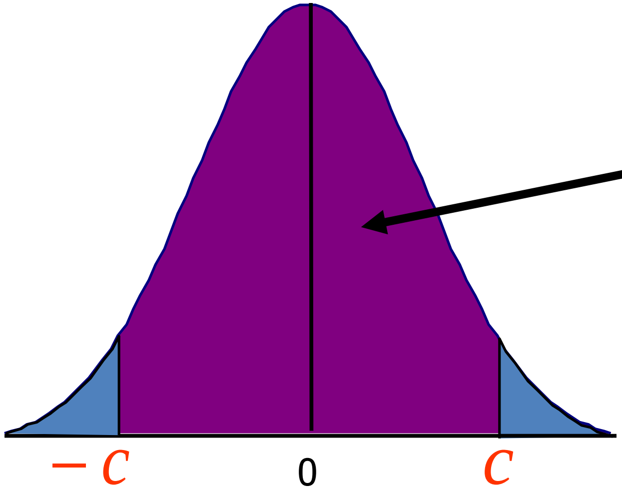
What is meant by "likely" ?

Suppose we have $Z \sim N(0,1)$

Then we can find c such that

$$P(-c < Z < c) = 0.95$$

From table 3 "Critical values for the standard normal distribution" we find: $c = 1.96$



95% of the standard normal observations are between −1.96 and 1.96: these are "likely" values.

Confidence intervals for μ when σ^2 is known

We apply this concept to the **Sampling Distribution** of $\bar{x}$

Recall: $\bar{x} \sim N(\mu, \sigma^2 / n)$ hence $\dfrac{\bar{x} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$

exactly true for normal data

otherwise approximately true when n>30 by CLT

Hence, $$P(-1.96 < \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} < 1.96) = 0.95$$

Or, by rearranging

$$P(\mu - 1.96 \frac{\sigma}{\sqrt{n}} < \bar{x} < \mu + 1.96 \frac{\sigma}{\sqrt{n}}) = 0.95$$

Confidence intervals for μ when σ^2 is known II

The picture can't be displayed.

Hence, 95% of the sample mean $\bar{x}$ are between $\mu \pm 1.96 \frac{\sigma}{\sqrt{n}}$

However: do **not** know μ

An alternative re-arrangement of the formula on the previous slide gives

$$(\bar{x} - 1.96 \frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96 \frac{\sigma}{\sqrt{n}})$$

is a 95% confidence interval for μ

We say we are 95% confident that the interval contains μ

Example

Suppose we know $\sigma^2 = (1/3)^2$ for a given set of normal data of size $n = 59$ and sample mean $\bar{x} = 63.59$

Then a 95% confidence interval for μ is given by

$$(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}})$$

i.e. $$63.59 \pm 1.96 \times \frac{1/3}{\sqrt{59}} = (63.505,\ 63.675)$$

i.e. we are 95% confident that the true parameter μ is contained in the interval $(63.505,\ 63.675)$

Confidence coefficient

We found a 95% confidence interval for the parameter μ hence, 95% of all confidence intervals will contain μ

95% is called the confidence coefficient

… other common choices are 90%, 99% etc

Let's have a look at the general case of a $\%100\times(1-\alpha)$ confidence interval for μ with known σ^2

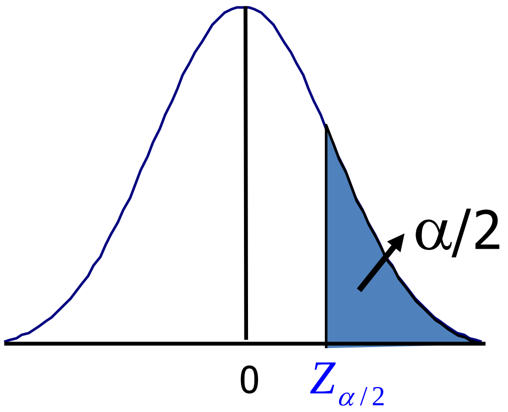
The general case

A $100 \times (1-\alpha)$ % confidence interval for μ with known σ^2 is given by

$$(\bar{x} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}})$$

Where $Z_{\alpha/2}$ depends on the confidence coefficient and is given by

$$P(Z > Z_{\alpha/2}) = \alpha/2$$



We can read the values from the Z score Table

Reading the critical values from the table

Let us try to read the values for $Z_{\alpha/2}$ from Table 3…

For a 95% confidence interval:

$$\alpha = 0.05 \qquad \alpha/2 = 0.025 \quad \Rightarrow \quad Z_{0.025} = 1.96$$

For a 90% confidence interval:

$$\alpha = 0.1 \qquad \alpha/2 = 0.05 \quad \Rightarrow \quad Z_{0.05} = 1.65$$

For a 99% confidence interval:

$$\alpha = 0.01 \qquad \alpha/2 = 0.005 \quad \Rightarrow \quad Z_{0.005} = 2.58$$

Remark

As smaller we choose α as 'more confident' we get that the interval contains the parameter μ. **But** at the same time the confidence interval gets wider and is therefore less precise.

The t-distribution

This question was addressed in 1908 when W.S. Gosset found that if we replace σ with the sample standard deviation s, the distribution becomes a t-distribution. If

$$T = \frac{\bar{x} - \mu}{s / \sqrt{n}}$$

then T has a t-distribution with n-1 degrees of freedom. The t-distribution is similar to the z-curve in that it is bell shaped, but the shape of the t-distribution changes with the degrees of freedom.

We will use the T-tables to get the critical t-values at different levels of α and degrees of freedom.

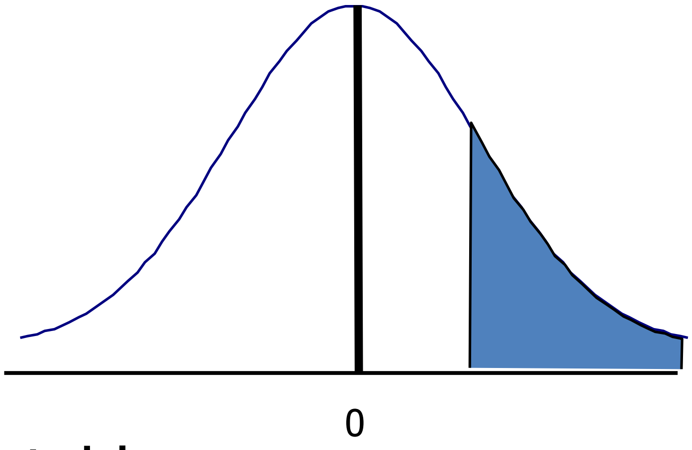
Properties of the t-distribution

a) Continuous and symmetric around 0
b) The shape of the t-distribution changes with the degrees of freedom. The t-distribution has heavier tails than the normal distribution.
c) t-distributions are indexed by the number of degrees of freedom. Example t(36) refers to a t-distribution with 36 degrees of freedom
d) **As the number of degrees of freedom increases, the t-distribution approaches the standard normal distribution**

Notation:

t(DF,α) = reads "t of alpha with DF degrees of freedom".

It is the value of t(DF) with $P(t(DF) > t) = \alpha$



Reading the t-table:

- Rows represent t values for a given DF
- Columns represent t values for a given α.

P-values

- “a statistical value that indicates the probability that the observed results are due to chance alone”
- benchmark of confidence in a result
- “the result was significant at $p<0.05$”
  - we could expect the observed result to occur by chance no more than 5 times in every 100
  - there is a 1:20 probability of this result occurring by chance
  - $p<0.05$ is arbitrary statistical significance
  - statistical significance does not of necessity mean clinical significance

t Distribution						
	α					
Degrees of freedom	.005 (one tail) .01 (two tails)	.01 (one tail) .02 (two tails)	.025 (one tail) .05 (two tails)	.05 (one tail) .10 (two tails)	.10 (one tail) .20 (two tails)	.25 (one tail) .50 (two tails)
1	63.657	31.821	12.706	6.314	3.078	1.000
2	9.925	6.965	4.303	2.920	1.886	.816
3	5.841	4.541	3.182	2.353	1.638	.765
4	4.604	3.747	2.776	2.132	1.533	.741
5	4.032	3.365	2.571	2.015	1.476	.727
6	3.707	3.143	2.447	1.943	1.440	.718
7	3.500	2.998	2.365	1.895	1.415	.711
8	3.355	2.896	2.306	1.860	1.397	.706
9	3.250	2.821	2.262	1.833	1.383	.703
10	3.169	2.764	2.228	1.812	1.372	.700
11	3.106	2.718	2.201	1.796	1.363	.697
12	3.054	2.681	2.179	1.782	1.356	.696
13	3.012	2.650	2.160	1.771	1.350	.694
14	2.977	2.625	2.145	1.761	1.345	.692
15	2.947	2.602	2.132	1.753	1.341	.691
16	2.921	2.584	2.120	1.746	1.337	.690
17	2.898	2.567	2.110	1.740	1.333	.689
18	2.878	2.552	2.101	1.734	1.330	.688
19	2.861	2.540	2.093	1.729	1.328	.688
20	2.845	2.528	2.086	1.725	1.325	.687
21	2.831	2.518	2.080	1.721	1.323	.686
22	2.819	2.508	2.074	1.717	1.321	.686
23	2.807	2.500	2.069	1.714	1.320	.685
24	2.797	2.492	2.064	1.711	1.318	.685
25	2.787	2.485	2.060	1.708	1.316	.684
26	2.779	2.479	2.056	1.706	1.315	.684
27	2.771	2.473	2.052	1.703	1.314	.684
28	2.763	2.467	2.048	1.701	1.313	.683
29	2.756	2.462	2.045	1.699	1.311	.683
Large (*z*)	2.575	2.327	1.960	1.645	1.282	.675

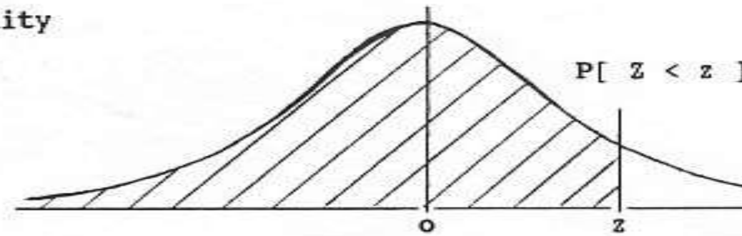
t-table Degrees of Freedom	Probability, p			
	0.1	0.05	0.01	0.001
1	6.31	12.71	63.66	636.62
2	2.92	4.30	9.93	31.60
3	2.35	3.18	5.84	12.92
4	2.13	2.78	4.60	8.61
5	2.02	2.57	4.03	6.87
6	1.94	2.45	3.71	5.96
7	1.89	2.37	3.50	5.41
8	1.86	2.31	3.36	5.04
9	1.83	2.26	3.25	4.78
10	1.81	2.23	3.17	4.59
11	1.80	2.20	3.11	4.44
12	1.78	2.18	3.06	4.32
13	1.77	2.16	3.01	4.22
14	1.76	2.14	2.98	4.14
15	1.75	2.13	2.95	4.07
16	1.75	2.12	2.92	4.02
17	1.74	2.11	2.90	3.97
18	1.73	2.10	2.88	3.92
19	1.73	2.09	2.86	3.88
20	1.72	2.09	2.85	3.85
21	1.72	2.08	2.83	3.82
22	1.72	2.07	2.82	3.79
23	1.71	2.07	2.82	3.77
24	1.71	2.06	2.80	3.75
25	1.71	2.06	2.79	3.73
26	1.71	2.06	2.78	3.71
27	1.70	2.05	2.77	3.69
28	1.70	2.05	2.76	3.67
29	1.70	2.05	2.76	3.66
30	1.70	2.04	2.75	3.65
40	1.68	2.02	2.70	3.55
60	1.67	2.00	2.66	3.46
120	1.66	1.98	2.62	3.37
infinity	1.65	1.96	2.58	3.29

STANDARD STATISTICAL TABLES

1. Areas under the Normal Distribution

The table gives the cumulative probability up to the standardised normal value z i.e.

$$P[\ Z < z\ ] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}Z^2)\ dZ$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5159	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7854
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8804	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9980	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

z	3.00	3.10	3.20	3.30	3.40	3.50	3.60	3.70	3.80	3.90
P	0.9986	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000